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On the Roots of Bessel- and P-Functions.

By Edward B. Van Vleck.

In the following article I shall confine my attention to the consideration of Bessel- and P-functions which are symmetrical in their properties with respect to the real axis of the complex variable. My first purpose is to give a proof of the theorem that between two successive positive or negative real roots of the Bessel function J_n there lies one and only one root of J_{n+1}. It is well known that at least one root of J_{n+1} is included in the interval, and it has been supposed that there is only one. But so far as I am aware, no proof of the latter point has been given, and I am confirmed in the impression by the following statement of Grey and Matthews in their recent "Treatise on Bessel Functions" (1895): "It seems probable that between every pair of successive real roots of $J_n(x)$ there is exactly one real root of $J_{n+1}(x)$. It does not appear that this has been strictly proved; there must in any case be an odd number of roots in the interval."

In section II a similar theorem is proved for contiguous ("benachbarte verwandte") Riemann P-functions. When two branches of such functions have a common group of substitutions, the roots of the one will alternate with those of the other in each of the three intervals between their singular points 0, 1, ∞. It is also pointed out that this theorem includes the foregoing theorem concerning Bessel functions as a limiting case. In a modified form it is extended in section III to contiguous P-functions with any number of branch-points. For a P-function with but three branch-points there are further formed in section II series of contiguous functions which may be used like the functions of Sturm to determine the number of its real roots between two given limits.

I.

1. *Between two successive positive or negative roots of the equation* $J_n = 0$ *there lies one and only one root of* $J_{n+1} = 0$. By n is here understood a real number.

When it is not an integer, the demonstration rests upon the relation

$$J_{n+1}J_{-n}+J_{-(n+1)}J_n=\frac{2\sin(n+1)\pi}{\pi x}. \tag{1}$$

In this case J_n and J_{-n} are independent solutions of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0, \tag{2}$$

whose singular points are 0 and ∞. From this it follows, by application of a theorem of Sturm to be cited later, that any two consecutive positive or consecutive negative roots of the equation $J_n=0$ are separated by one root of $J_{-n}=0$ and only one

The same may be also proved in a very simple manner from the formula

$$\frac{J_{-n}}{J_n}=-\int\frac{2\sin n\pi.dx}{\pi x\,J_n^2}. \tag{3}$$

For this shows that $\frac{d\frac{J_{-n}}{J_n}}{dx}$ is of constant sign for all positive or all negative values of x. Therefore $\frac{J_{-n}}{J_n}$ either continually increases or continually decreases as x passes from 0 to ∞. It follows that x passes alternately through the zeros and infinities of this quotient; that is, the real roots of J_n and J_{-n} succeed each other alternately.

Let now any real root of J_n be substituted in (1). It then becomes

$$J_{n+1}J_{-n}=\frac{2\sin(n+1)\pi}{\pi x}.$$

This shows that J_{n+1} and J_{-n} both change sign an even number of times or both an odd number of times between two consecutive roots of J_n. But it has just been demonstrated that J_{-n} has one and only one root in the interval, and hence changes but once. Therefore between two consecutive real roots of J_n there lies an odd number of roots of J_{n+1}. The substitution of a real root of J_{n+1} will prove in like manner that between two consecutive roots of J_{n+1} there lies an odd number of roots of J_n. These two results can be reconciled only when the roots of J_n and J_{n+1} alternate between $x=0$ and $x=\infty$, which was to be proved.

2. When n is an integer, the above proof fails. For J_n and J_{-n} are no longer independent integrals of the equation (2), but $J_{-n} = (-1)^n J_n$; likewise $J_{-(n+1)} = (-1)^{n+1} J_{n+1}$. At the same time $\sin (n+1)\pi = 0$, and both members of (1) therefore vanish identically. To provide for this exceptional case we will introduce the Bessel functions of the second kind, Y_n and Y_{n+1}, in place of J_{-n} and $J_{-(n+1)}$. The relation

$$Y_n J_{n+1} - Y_{n+1} J_n = \frac{1}{x}$$

has then only to be substituted for (1) and the proof is as before. Our theorem is therefore true for all real values of n.

II.

3. Two functions $P\begin{pmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{pmatrix}$ and $P_1\begin{pmatrix} 0 & \infty & 1 & \\ \alpha_1 & \beta_1 & \gamma_1 & x \\ \alpha_1' & \beta_1' & \gamma_1' & \end{pmatrix}$ are said to be *related* when the differences of corresponding exponents, $\alpha - \alpha_1$, $\alpha' - \alpha_1'$, etc., are integers, positive or negative. Place

$$\lambda = \alpha' - \alpha, \quad \mu = \beta' - \beta, \quad \nu = \gamma' - \gamma,$$

and denote by λ_1, μ_1, ν_1 the corresponding exponent-differences of P_1. Since the sum of the six exponents of either function is unity, it follows that $\Sigma |\lambda_1 - \lambda|$ —that is, $|\lambda_1 - \lambda| + |\mu_1 - \mu| + |\nu_1 - \nu|$—is an even number. If it is equal to 2, the two functions are said to be *contiguous.*

Taking now two related P-functions, and employing the usual notation P^{α}, $P^{\alpha'}$, $P_1^{\alpha_1}$, $P_1^{\alpha_1'}$ for the four fundamental branches which correspond to the exponents α, α', α_1, α_1', we have according to Riemann the relation

$$(P^{\alpha} P_1^{\alpha_1'} - P^{\alpha'} P_1^{\alpha_1}) x^{-\bar{\alpha}} (1-x)^{-\bar{\gamma}} = R(x), \tag{4}$$

in which $\bar{\alpha}$ denotes the smaller of the two numbers $\alpha + \alpha_1'$ and $\alpha' + \alpha_1$, $\bar{\gamma}$ the smaller of the two numbers $\gamma + \gamma_1'$ and $\gamma' + \gamma_1$, and $R(x)$ is a polynomial.* The degree of the polynomial is $\frac{\Sigma |\lambda_1 - \lambda|}{2} - 1$. We shall here confine ourselves to the consideration of contiguous functions, and the polynomial is therefore to be replaced by a constant.

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*See Riemann's collected works, p. 73-4.

In place of the four particular branches used in the last equation, four others, more general in character, may be introduced. The two branches of P may be selected arbitrarily and written in the form

$$y = c_1 P^{\alpha} + c_2 P^{\alpha'}, \ \bar{y} = c_3 P^{\alpha} + c_4 P^{\alpha'},$$

where c_1, c_2, c_3 and c_4 are constants. For the branches of P_1 we will take

$$y_1 = c_1 P_1^{\alpha_1} + c_2 P_1^{\alpha_1'}, \ \bar{y}_1 = c_3 P_1^{\alpha_1} + c_4 P_1^{\alpha_1'}.$$

y_1 and $\bar{y}_1$ will then have the same group of substitutions as y and $\bar{y}$ respectively. Between the four branches there exists the obvious relation

$$y\bar{y}_1 - \bar{y}y_1 = (c_1 c_4 - c_2 c_3)(P^{\alpha} P_1^{\alpha_1'} - P^{\alpha'} P_1^{\alpha_1}),$$

whence

$$y\bar{y}_1 - \bar{y}y_1 = Cx^{\bar{\alpha}}(1 - x)^{\bar{\gamma}}. \tag{5}$$

4. This equation is obviously analogous to (1). When the exponents and the constants c are real, the functions y, y_1, $\bar{y}$, $\bar{y}_1$ are symmetrical in their properties with respect to the real axis of x. The equation may then be used to investigate the relative position of the roots of y and y_1. First we observe that y and $\bar{y}$, as branches of the same P-function, are solutions of a common linear differential equation of the second order with the singular points 0, 1, ∞. Sturm, in his classic memoir in the first volume of Liouville's Journal, p. 106, has proved that the real roots of any two symmetrical solutions of a linear differential equation of the second order with real coefficients will, under certain specified restrictions, alternate with one another. These restrictions, stated in the language of the Theory of Functions of a complex variable, confine the application of the theorem to an interval of the real axis included between consecutive singular points of the differential equation. To apply it to the case before us, the x-axis is divided into three intervals $\infty 0$, $0\ 1$, $1\ \infty$, in each of which a pair of consecutive roots of y will be separated by a single root of $\bar{y}$. Let now a root of y be substituted in (5). We have

$$-\bar{y}y_1 = Cx^{\bar{\alpha}}(1 - x)^{\bar{\gamma}}.$$

In this the right-hand member has a constant sign throughout each interval. Hence between two consecutive roots of y in each interval, $\bar{y}$ and y_1 must change sign either both an even number of times or both an odd number of times. But $\bar{y}$ changes sign once and only once. It follows that an odd number of roots of y_1 lie between the two roots of y. Since the same is obviously true when y_1 and y are interchanged, we have the theorem:

The real roots of two contiguous P-functions which are symmetrical in their properties with respect to the real axis of the complex variable, and which have the same group of substitutions, will alternate with each other in each interval of the x-axis included between their singular points 0, 1 *and* ∞.

5. With the aid of this theorem we can derive a series of Sturm functions for the determination of the number of real roots of y between two limits a and b. The only restriction upon a and b is that no singular points shall be included between them. Let first some function y_n be selected which is related to y and has no roots between a and b. It will be proved later that such a selection is always possible. Between the two functions, y, y_n, any continuous succession of contiguous functions, y_1, y_2, etc., are next to be inserted. Then in the series y, y_1, $\pm y_2$, $\pm y_3$, ... $\pm y_n$, the signs can be so fixed that it shall have, between a and b, all the properties of Sturm functions. These are as follows:

(1). When x is placed equal to a root of any of these functions except the first and last, the two contiguous functions have opposite signs.

(2). When x traverses the interval from a to b and passes through a root of y, the sign of $\frac{y}{y_1}$ changes always from — to + or always from + to —.

(3). y_n does not change sign between a and b.

The last of these properties has been ensured by our selection of y_n. The second follows from the fact that the roots of y and y_1 alternate in the interval between a and b. The first can be effected by the proper determination of the signs placed before the successive functions of the series. For this purpose let us consider the relations

$$\left.\begin{aligned} y_i \quad \bar{y}_{i-1} - \bar{y}_i \quad y_{i-1} &= C_i x^{\bar{\alpha}_i} (1 - x)^{\bar{\beta}_i}, \\ y_{i+1} \bar{y}_i \quad - \bar{y}_{i+1} y_i \quad &= C_{i+1} x^{\bar{\alpha}_{i+1}} (1 - x)^{\beta_{i+1}}. \end{aligned}\right\} \tag{6}$$

The constant C in either of these equations, if not already known, can be determined by various methods; for example, by expanding each member of the equation into a series arranged according to powers of x and comparing the leading coefficients of the two series. The right-hand members may therefore be regarded as known. Let us now place x equal to some root of y_i. We have then

$$-\frac{y_{i-1}}{y_{i+1}} = \frac{C_i}{C_{i+1}} x^{\bar{\alpha}_i - \bar{\alpha}_{+1}} (1 - x)^{\bar{\beta}_i - \bar{\beta}_{i+1}}. \tag{7}$$

Here the right-hand member does not change sign between a and b. If it is positive, y_{i-1} and y_{i+1} have opposite signs whenever x passes through a root of y_i; if negative, they have the same sign. In the first case, to secure the desired property for the series, it is only necessary to use the same sign before $\bar{y}_{i+1}$ as before y_{i-1}; in the second case, the opposite sign.

The signs can also be determined without the direct use of equations (6), if the recurring relation between three consecutive functions

$$y_{i+1} = m_i y_i - n_i y_{i-1} \tag{8}$$

is known. The existence of this relation was proved by Riemann, and may be deduced from the equations (6). It is obvious that n_i is equal to the right-hand member of (7), and may therefore be employed to discriminate in regard to the signs.

It remains finally to prove that there are functions y_n which are related to y and have no roots between a and b. This may be easily inferred from Professor Klein's investigation, "Ueber die Nullstellen der hypergeometrischen Reihe" in the 37th volume of the Mathematische Annalen. The expression there given for the number of roots of a P-function in an interval between two singular points whose exponent-differences are μ and ν is $E\left(\frac{\lambda - \mu - \nu + 1}{2}\right) + \varepsilon$. $E(p)$ denotes the largest positive integer contained in p, and is zero when p is negative; ε is either zero or unity. If, then, the sum of two exponent-differences is greater than the third, there is at most a single root in the interval between the two corresponding singular points. From the same article it is also clear that this root, if there be one, can be removed by increasing the two exponent-differences each by unity (i. e. by the "lateral attachment" of a half plane to the triangle which he employs in his investigation). Now the exponent-differences may be so selected for y_n as to meet these conditions for that interval which includes a and b. For the differences of corresponding exponent-differences of y_n and y (viz. $\lambda_n - \lambda$, $\mu_n - \mu$, $\nu_n - \nu$) are integers. Since the exponent-sum $\Sigma(\alpha)$ for both functions is unity, the sum of these three differences must be an even integer. Subject to this restriction λ_n, μ_n, ν_n are at our disposal. They may accordingly be chosen in the manner just specified, and in an infinite number of ways.

6. The existence of our Sturm functions to determine the number of roots of our P-function between a and b has thus been established. When y_n has been selected, it is immaterial what chain of intermediate functions is used to connect it with y. As was pointed out in my article in the 16th volume of this journal, the exponent-differences of two contiguous functions may be related to each other in any one of three ways. Either first, two corresponding exponent-differences may be the same in both functions, and the third pair differ by 2; or secondly, two exponent-differences of the one function may exceed the corresponding two of the other by a unit, the third pair being equal; or lastly, one exponent-difference may be a unit greater and a second a unit smaller than the corresponding exponent-differences of the other function, the third pair again being equal. Corresponding to these three possibilities there are three types of *regular* series of contiguous functions; either a single exponent-difference increases (or decreases) regularly by two units from function to function; or secondly, two exponent-differences increase each by a unit; or lastly, one exponent-difference increases while a second decreases each by a unit.

A particular case of the series here discussed is found in an article by Hurwitz in the 38th volume of the Mathematische Annalen, and was used by him to determine the number of roots of the hypergeometric series between 0 and 1. The series there employed was $F(\alpha, \beta, \gamma, x)$,

$$F(\alpha+1, \beta+1, \gamma+1, x), \ldots . \pm F(\alpha+n, \beta+n, \gamma+n, x),$$

n being sufficiently large to ensure that the last function has no roots between 0 and 1. This series is of the 2d type above mentioned. The signs placed before the functions Hurwitz determined by the recurring relation (8).

7. In concluding this section it may be mentioned that the results of section I are included as a limiting case under those given here. J_n and J_{-n} are the limits of the two fundamental branches P_n and P_{-n} of $P\begin{pmatrix} 0 & \infty & a & \\ +n & \beta_1 & \gamma_1 & x \\ -n & \beta_2 & \gamma_2 & \end{pmatrix}$ when under proper restrictions a, β_1, β_2, γ_1 and γ_2 become infinite and disappear from the differential equation.* The point at infinity then becomes an essential singularity, and the only remaining parameters of the function are the

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*See Craig's Linear Differential Equations, p. 193.

exponents ± n which belong to the origin. Accordingly J_n and J_{n+1}, also J_{-n} and $J_{-(n+1)}$, are to be considered as corresponding branches of two contiguous Bessel functions. Therefore the relation (4) passes over into (1), and the theorem concerning the alternation of the roots of J_n and J_{n+1} is thus seen to be a limiting case of the theorem given in §4.

III.

8. Riemann's P-function with three singular points satisfies the differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1}\right)\frac{dy}{dx} + \left(\frac{-\alpha\alpha'}{x} + \frac{\gamma\gamma'}{x-1} + \beta'\beta\right)\frac{y}{x(x-1)} = 0.$$

The general form of a regular linear differential equation of the second order with any number of singular points is

$$\frac{d^2y}{dx^2} + \sum_{i=1}^{i=n}\left(\frac{1-\alpha_i-\alpha_i'}{x-e_i}\right)\frac{dy}{dx} + \left(\frac{\alpha_1\alpha_1'\prod\limits_{i=2}^{i=n}(e_1-e_i)}{x-e_1} + \dots + \frac{\alpha_n\alpha_n'\prod\limits_{i=1}^{i=n-1}(e_n-e_i)}{x-e_n}\right.$$
$$\left. + \alpha_\infty\alpha_\infty' x^{n-2} + A_1 x^{n-3} + \dots A_{n-2}\right)\frac{y}{\Pi(x-e_i)} = 0.$$

This differs from the preceding by having $n-2$ constants $A_1, \dots A_{n-2}$, which are independent of the singular points and their exponents. These constants are called by Klein "*accessory parameters.*" By a generalized P-function we will understand a solution of this differential equation, and we will denote it in the customary manner by $P\begin{pmatrix} e_1 & \dots & e_n & \infty & & \\ \alpha_1 & \dots & \alpha_n & \alpha_\infty & A_1 \dots A_{n-2}, & x \\ \alpha_1' & & \alpha_n' & \alpha_\infty' & & \end{pmatrix}$. *Two P-functions shall by definition be related when the groups of substitutions for any two corresponding branches are the same and are due to circuits of the variable x around common branch-points $e_1, \dots e_r$.* Corresponding exponents in the two functions therefore differ by integers. But this, though necessary, is by no means a sufficient condition as before in the case of functions with but three branch-points. The group of the function depends upon its accessory parameters as well as upon its exponents, and the former must therefore be properly determined in both functions. But unless the total number of singular points is greater than r, the number of parameters is not sufficiently large to permit of a common group. To increase the number of parameters, I used in the article already mentioned certain accessory singular points* which were designated by ρ_1, ρ_2, etc. The exponents of each of these points are fixed, having the values 2 and 0, and consequently a circuit of x around one or more of these points will not give rise to a substitution of the group. On the other hand, the location of these points for each function is arbitrary, thus placing the required additional constants at our disposal. The least number of such points is in general $r - 2$. The complete symbols for two related functions are therefore

$$P\begin{pmatrix} e_1 \dots\dots e_r & \infty & \\ \alpha \qquad\quad \nu & \xi & \rho_1, \rho_2, \dots\dots \rho_{r-2}, A_1, \dots\dots A_{r-2}, x \\ \alpha' \qquad\quad \nu' & \xi' & \end{pmatrix}$$

and

$$P_1\begin{pmatrix} e_1 \dots\dots e_r & \infty & \\ \alpha_1 \qquad\quad \nu_1 & \xi_1 & \rho_1', \rho_2', \dots\dots \rho_{r-2}', A_1', \dots\dots A_{r-2}', x \\ \alpha_1' \qquad\quad \nu_1' & \xi_1' & \end{pmatrix},$$

the differences of corresponding exponents being integers.

The possibility of such a determination of the accessory points and parameters in the two functions that they will have a common group is here assumed and may be inferred from the fact that the number of parameters in the function is equal to the number of conditions imposed by this requirement. A rigorous demonstration was given in the article last cited for the case where one and hence both functions are defined by equations which permit of algebraic integrals. A strict proof is still needed for the general case.

9. Riemann's relation between related P-functions may be extended so as to apply to the functions under consideration. His proof of the relation presupposed only the existence of a common group of substitutions for the two functions, due to circuits of x around common branch-points. The same reasoning, extended to the case of more than three branch-points, will give

$$P^{\alpha} P_1^{\alpha_1'} - P^{\alpha'} P_1^{\alpha_1} = (x - e_1)^{\bar{\alpha}} \dots\dots (x - e_r)^{\bar{\nu}} R(x),$$

in which $R(x)$ is a polynomial. It will be noticed that the accessory points do not enter into the right-hand member of this equation. Two related functions will again be said to be contiguous when the sum of the absolute values of the differences of the corresponding-exponent-differences, $\Sigma|\lambda_i' - \lambda_i|$, is equal to 2. For contiguous functions the polynomial, being of degree $\frac{\Sigma|\lambda_i' - \lambda_i|}{2} - 1$, reduces to a constant. If now y, $\bar{y}$, y_1, $\bar{y}_1$ be defined as in §3, but in terms of these new functions, we have

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* These points were there designated as *ganzzahlige Punkte* (p. 17). In his lectures of 1893-4 Klein has since called them *Nebenpunkte*, which I have here translated into *accessory points*.

$$y\bar{y}_1 - \bar{y}y_1 = C(x-e_1)^{\bar{\alpha}} \dots (x-e_n)^{\bar{\nu}}. \tag{9}$$

10. This last equation can now be employed to investigate the relative position of the roots of two corresponding branches of any two contiguous functions, y and y_1, the properties of which are symmetrical with respect to the real axis of the variable x. The branch-points e, the exponents, and accessory parameters are then real, while the accessory points are either real or symmetrically located with respect to the axis. If neither function has an accessory point in the interval $e_i e_{i+1}$ between two of their common branch-points, the alternation of the roots of y with those of y_1 throughout the whole interval may be argued as before with the help of the above-mentioned theorem of Sturm. Should, however, either function have accessory points between e_i and e_{i+1}, we can go no further than to infer the alternation of the roots of y and $\bar{y}$ for the interval between e_i and the nearest accessory point of either function.

To bridge the entire interval between e_i and e_{i+1}, it is necessary to use in place of Sturm's theorem the well-known relation between two solutions of the equation $y'' + py' + q = 0$, viz.:

$$y\frac{d\bar{y}}{dx} - \bar{y}\frac{dy}{dx} = e^{-\int p \, . \, dx}$$

For the case before us

$$p = \frac{1-\alpha-\alpha'}{x-e_1} + \dots + \frac{1-\nu-\nu'}{x-e_r} - \frac{1}{x-\rho_1} \dots - \frac{1}{x-\rho_{r-2}},$$

whence follows

$$y\frac{d\bar{y}}{dx} - \bar{y}\frac{dy}{dx} = (x-e_1)^{\alpha+\alpha'-1} \dots (x-e_r)^{\nu+\nu'-1}(x-\rho_1) \dots (x-\rho_{r-2}).$$

Here it is evident that for values of x close to the two extremities of the interval $e_i e_{i+1}$, the right-hand member will have unlike or like signs according as there is an odd or an even number of accessory points between e_i and e_{i+1}. In the first case we find, by supposing y to vanish in the equation, that two roots of y

are separated by an even number of roots of y; in the second case, they are separated by an odd number of roots of $\overline{y}$.

We return now to equation (9) and substitute in this also a root of y. The equation then shows that $\overline{y}$ and y_1 have each an odd number of roots or each an even number of roots between the two roots of y. Combining this result with that of the last paragraph we obtain our final theorem:

Given two branches of contiguous generalized P-functions which have a common group of substitutions due to circuits of the variable around common branch-points $e_1, \ldots . e_r$; *if the two functions are symmetrical with respect to the real axis of the variable, two consecutive real roots of either function in the interval between* e_i *and* e_{i+1} *will be separated by an even or an odd number of roots of the contiguous function, according as the two roots of the function include between them an odd or an even number of its accessory points.*